Universal Formulation for the Perturbed Two-Body Problem

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The theory and applications of the uniform treatment of Kepler motion are extended to the case where a known perturbing force is present. The cases where the perturbing force is either coplanar or noncoplanar with the instantaneous plane of motion of the perturbed body are considered. In several applications, the accuracy and efficiency of this extended method for the calculation of ephemerides of the perturbed body are shown.

I. Introduction

THE essential object of this article is to extend the theory and applications of the uniform treatment of Kepler motion to the case where a perturbing force is present. In such a case, the vectorial equation of motion of the perturbed body is

$$\ddot{\boldsymbol{r}} + k^2 \frac{\boldsymbol{r}}{r^3} = \boldsymbol{P} \tag{1}$$

with the initial conditions

$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0 \qquad \dot{\boldsymbol{r}}(t_0) = \dot{\boldsymbol{r}}_0 \tag{2}$$

where $k = 0.01720209895$ is the Gaussian gravitational constant, $\boldsymbol{P}$ is a given perturbing acceleration, and $\boldsymbol{r}_0$ and $\dot{\boldsymbol{r}}_0$ are given vectors of initial position and velocity.

It is important to remark that the only requirement about the perturbing function $\boldsymbol{P}$ is to have a clear definition of it in order to be able to calculate its value and that of its successive derivatives either analytically or numerically. This is an advantage of our method because no further difficulties should appear in problems involving non-point-mass potential, nonconservative perturbations, and so forth. On the other hand, the difficulties that would appear from large truncation errors would be the same in our method as well as in conventional methods of numerical integration.

The theory of the uniform treatment of the Keplerian (unperturbed) motion was developed by Stumpff,[1] who arrived at a solution without using the classical orbit elements and without variations owing to the different types of conic section orbits. Also, Herrick[2] further developed Stumpff's ideas by proposing the use of "universal" variables and formulas.

Here, Stumpff's development is modified to take into account the effects of the perturbation $\boldsymbol{P}$. For the particular case where the motion of the perturbed and perturbing bodies are coplanar, the position and velocity vectors of the perturbed body for any instant t are obtained in the customary form

$$\boldsymbol{r} = F\boldsymbol{r}_0 + G\dot{\boldsymbol{r}}_0 \tag{3}$$

$$\dot{\boldsymbol{r}} = \dot{F}\boldsymbol{r}_0 + \dot{G}\dot{\boldsymbol{r}}_0 \tag{4}$$

where F, G, $\dot{F}$, and $\dot{G}$ are scalar functions of t and the local elements at t_0. For the general case of noncoplanarity of both orbits, we add to the right-hand members of Eqs. (3) and (4) further terms $H\boldsymbol{g}_0$ and $\dot{H}\boldsymbol{g}_0$, respectively, where H and $\dot{H}$ are again scalar functions and $\boldsymbol{g}_0$ is the angular momentum vector $\boldsymbol{r}_0 \times \dot{\boldsymbol{r}}_0$, orthogonal to the instantaneous plane of motion of the perturbed body.

Our formulation provides a method for calculating ephemerides in a precise and efficient way appropriate to the particular problem at hand, as compared to a general numerical integration method. It is well known that the standard methods of numerical integration lose their efficiency and accuracy when the bodies involved come close to collision. It has been found that this difficulty may be minimized by submitting the differential equations to a regularizing transformation before performing their numerical integration.[3,4] The method we are proposing is a combination of analytical and numerical procedures not requiring any process of numerical integration, and it is rather insensitive to close approaches between the involved bodies.

Finally, in several examples we have tested the efficiency of our method as compared with analytical solutions in the Lagrangian case of three bodies and with a numerical method of integration of ordinary differential equations in the general noncoplanar case.

II. Notations and Definitions

For any instant t we have

$$r^2 = x^2 + y^2 + z^2 \tag{5}$$

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{6}$$

and

$$(\boldsymbol{r} \cdot \dot{\boldsymbol{r}}) = r\dot{r} = \; = x\dot{x} + y\dot{y} + z\dot{z} \tag{7}$$

By using a unit of time equal to $1/k$ and assuming a negligible mass for the perturbed body, its equation of motion [Eq. (1)] reduces to

$$\ddot{\boldsymbol{r}} + \frac{\boldsymbol{r}}{r^3} = \boldsymbol{P} \tag{8}$$

If the perturbation $\boldsymbol{P}$ is due to the attraction of a body of mass m_P and position vector $\boldsymbol{R}$, we have

$$\boldsymbol{P} = m_P \left\{ \frac{\boldsymbol{R} - \boldsymbol{r}}{|\boldsymbol{R} - \boldsymbol{r}|^3} - \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3} \right\} \tag{9}$$

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The local fundamental invariables defined by Stumpff are the functions

$$\mu = \frac{1}{r^3} \tag{10a}$$

$$\sigma = \frac{\dot{r}}{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r^2} \tag{10b}$$

$$\omega = \frac{V^2}{r^2} \tag{10c}$$

which do not vary in a transformation of coordinates, depending only on the position $\boldsymbol{r}$ and velocity $\dot{\boldsymbol{r}}$ relative to the central body. In fact, the actual invariants in different coordinate systems are the modulus of the vectors $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ and the angle between them.

From the fundamental invariables, one obtains the derived invariables

$$\varepsilon = \omega - \mu \tag{11a}$$

$$\rho = 2\mu - \omega \tag{11b}$$

$$\theta = \omega - \sigma^2 \tag{11c}$$

Our next purpose is to obtain time derivatives of the fundamental and derived invariables. Thus, due to the presence of the perturbing force, we find

$$\dot{\mu} = -\frac{3(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})}{r^5} \tag{12a}$$

$$\dot{\sigma} = \frac{|\dot{\boldsymbol{r}}|^2}{r^2} - \frac{1}{r^3} - \frac{2(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})^2}{r^4} + \frac{(\boldsymbol{r} \cdot \boldsymbol{P})}{r^2} \tag{12b}$$

$$\dot{\omega} = -\frac{2(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})}{r^2}\left\{\frac{1}{r^3} + \frac{|\dot{\boldsymbol{r}}|^2}{r^2}\right\} + \frac{2(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r^2} \tag{12c}$$

or

$$\dot{\mu} = -3\mu\sigma \tag{13a}$$

$$\dot{\sigma} = \omega - \mu - 2\sigma^2 + \frac{(\boldsymbol{r} \cdot \boldsymbol{P})}{r^2} \tag{13b}$$

$$\dot{\omega} = -2\sigma(\mu + \omega) + \frac{2(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r^2} \tag{13c}$$

In a similar way, we obtain for the derived invariables

$$\dot{\varepsilon} = -\sigma(2\omega - \mu) + \frac{2(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r^2} \tag{14a}$$

$$\dot{\rho} = -2\rho\sigma - \frac{2(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r^2} \tag{14b}$$

$$\dot{\theta} = -4\theta\sigma + \frac{2(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r^2} - \frac{2\sigma(\boldsymbol{r} \cdot \boldsymbol{P})}{r^2} \tag{14c}$$

Obviously, all of these expressions reduce for $\boldsymbol{P} \equiv 0$ to those obtained by Stumpff for the unperturbed Keplerian motion.

From the definition (10b), we have $\dot{r} = \sigma r$ and, by differentiating twice, we obtain

$$\ddot{r} = r(\varepsilon - \sigma^2) + \frac{(\boldsymbol{r} \cdot \boldsymbol{P})}{r} \tag{15}$$

$$\dddot{r} = \dot{r}(\varepsilon - \sigma^2) + r\dot{\varepsilon} - 2r\sigma\dot{\sigma} + \frac{(\dot{\boldsymbol{r}} \cdot \boldsymbol{P})}{r} + \frac{(\boldsymbol{r} \cdot \dot{\boldsymbol{P}})}{r} - \frac{\dot{r}(\boldsymbol{r} \cdot \boldsymbol{P})}{r^2} \tag{16}$$

By eliminating σ and $(\varepsilon - \sigma^2)$ and replacing $\dot{\sigma}$ and $\dot{\varepsilon}$ by Eqs. (13b) and (14a), we obtain

$$\dddot{r} + \frac{3\dot{r}\ddot{r}}{r} + \frac{\dot{r}}{r^3} = \frac{1}{r}[3(\dot{\boldsymbol{r}} \cdot \boldsymbol{P}) + (\boldsymbol{r} \cdot \dot{\boldsymbol{P}})] \tag{17}$$

which is a third-order differential equation for the magnitude of the position vector of the perturbed body. For $\boldsymbol{P} \equiv 0$, the right-hand member reduces to zero and is in agreement with Stumpff's result.

III. Introduction of a Regularizing Variable

To simplify the differential equation (17), we again follow Stumpff's development by introducing Sundman's[8] regularizing variable q such that

$$\dot{q} = \frac{dq}{d\tau} = \frac{1}{r} \tag{18}$$

or

$$\tau = \int_{\delta}^{q} r(q)\, dq \tag{19}$$

where $\tau = k(t - t_0)$ for the initial epoch $\tau = 0$, $q = 0$.

With derivatives with respect to q indicated by primes, Eq. (17) becomes

$$r''' + \alpha^2 r' = r[3(\boldsymbol{r}' \cdot \boldsymbol{P}) + (\boldsymbol{r} \cdot \boldsymbol{P}')] \tag{20}$$

where

$$\alpha^2 = \frac{1 - r''}{r} = \rho r^2 - (\boldsymbol{r} \cdot \boldsymbol{P}) \tag{21}$$

and it can be shown that

$$\frac{d}{dq}(\alpha^2) = -[3(\boldsymbol{r}' \cdot \boldsymbol{P}) + (\boldsymbol{r} \cdot \boldsymbol{P}')] \tag{22}$$

In the unperturbed case, the regularizing transformation leads to Eq. (20), with the right-hand member equal to zero. This fact suggests the possibility of applying, for the perturbed case, the method of the variation of parameters. However, this procedure leads to a complicated process scarcely useful in practice.

IV. Formal Application of Stumpff's Functions

The modulus of the position vector, as a function of q, admits a Taylor expansion in powers of q

$$r(q) = r_0 + \frac{r_0'}{1!}q + \frac{r_0''}{2!}q^2 + \frac{r_0'''}{3!}q^3 + \frac{r_0^{IV}}{4!}q^4 + \frac{r_0^{V}}{5!}q^5 + \cdots \tag{23}$$

where r_0, r_0', r_0'',..., are values corresponding to $t = t_0$ or $q = 0$.

Stumpff's functions are defined by

$$c_n(\lambda^2) = \sum_{k=0}^{\infty} \frac{(-\lambda^2)^k}{(2k + n)!} \tag{24}$$

with the recurrent relationship

$$c_n + \lambda^2 c_{n+2} = \frac{1}{n!} \tag{25}$$

For the present application, we adopt

$$\lambda^2 = \alpha_0^2 q^2 = [\rho_0 r_0^2 - (\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)]q^2 \tag{26}$$

where α_0 is the value of α at $t = t_0$. Then it is possible to prove that the function $c_n(\alpha_0^2 q^2)$ satisfy the differential relationship

$$\frac{d}{dq}[c_{n+1}q^{n+1}] = c_n q^n \tag{27}$$

By virtue of Eq. (25), Eq. (23) becomes

$$r(q) = r_0 + r_0'(c_1 + \lambda^2 c_3)q + r_0''(c_2 + \lambda^2 c_4)q^2 + r_0'''(c_3 + \lambda^2 c_5)q^3 + r_0^{IV}(c_4 + \lambda^2 c_6)q^4 + r_0^V(c_5 + \lambda^2 c_7)q^5 + \cdots \tag{28}$$

Replacing λ^2 by expression (26), we get

$$r(q) = r_0 + c_1 r_0' q + c_2 r_0'' q^2 + c_3(r_0''' + \alpha_0^2 r_0')q^3 + c_4(r_0^{IV} + \alpha_0^2 r_0'')q^4 + c_5(r_0^V + \alpha_0^2 r_0''')q^5 + \cdots \tag{29}$$

Now, by virtue of Eqs. (20) and (22) and the expressions for the invariables of Sec. II, we obtain, after some manipulations

$$r(q) = r_0 + c_1\sigma_0 r_0^2 q + c_2\varepsilon_0 r_0^3 q^2 + S(q) \tag{30}$$

where

$$\begin{aligned} S(q) &= c_2(\mathbf{r}_0 \cdot \mathbf{P}_0)r_0 q^2 + c_3\{r_0[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')]\}q^3 \\ &+ c_4\{2r_0'[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')] + r_0[3(\mathbf{r}_0'' \cdot \mathbf{P}_0) + 4(\mathbf{r}_0' \cdot \mathbf{P}_0') \\ &+ (\mathbf{r}_0 \cdot \mathbf{P}_0'')]\}q^4 + c_5\{3r_0''[3(\mathbf{r}_0' \cdot \mathbf{P}_0 + (\mathbf{r}_0 \cdot \mathbf{P}_0'))] \\ &+ 3r_0'[3(\mathbf{r}_0'' \cdot \mathbf{P}_0) + 4(\mathbf{r}_0' \cdot \mathbf{P}_0') + (\mathbf{r}_0 \cdot \mathbf{P}_0'')] + r_0[3(\mathbf{r}_0''' \cdot \mathbf{P}_0) \\ &+ 7(\mathbf{r}_0'' \cdot \mathbf{P}_0') + 5(\mathbf{r}_0' \cdot \mathbf{P}_0'') + (\mathbf{r}_0 \cdot \mathbf{P}_0''')]\}q^5 + \cdots \end{aligned} \tag{31}$$

In the unperturbed case, obviously, $S(q)$ reduces to zero. It is important to remark that in the actual calculations it was necessary to extend this series up to the fifth power of q.

V. Main Equation for the Perturbed Problem

Introducing in the integrand of Eq. (19) the expression (30) for $r(q)$, and by virtue of Eq. (27), we obtain

$$\begin{aligned} \tau &= r_0 q + c_2\sigma_0 r_0^2 q^2 + c_3\varepsilon_0 r_0^3 q^3 + c_3(\mathbf{r}_0 \cdot \mathbf{P}_0)r_0 q^3 \\ &+ c_4\{r_0[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')]\}q^4 + c_5\{2r_0'[3(\mathbf{r}_0' \cdot \mathbf{P}_0) \\ &+ (\mathbf{r}_0 \cdot \mathbf{P}_0')] + r_0[3(\mathbf{r}_0'' \cdot \mathbf{P}_0) + 4(\mathbf{r}_0' \cdot \mathbf{P}_0') + (\mathbf{r}_0 \cdot \mathbf{P}_0'')]\}q^5 \\ &+ c_6\{3r_0''[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')] + 3r_0'[3(\mathbf{r}_0'' \cdot \mathbf{P}_0) \\ &+ 4(\mathbf{r}_0' \cdot \mathbf{P}_0') + (\mathbf{r}_0 \cdot \mathbf{P}_0'')] + r_0[3(\mathbf{r}_0''' \cdot \mathbf{P}_0) \\ &+ 7(\mathbf{r}_0'' \cdot \mathbf{P}_0') + 5(\mathbf{r}_0' \cdot \mathbf{P}_0'') + (\mathbf{r}_0 \cdot \mathbf{P}_0''')]\}q^6 + \cdots \end{aligned} \tag{32}$$

Thus, we obtain from Eq. (32) the main equation for the perturbed problem in the form

$$1 - \phi(z) = z + c_2\eta_0 z^2 + c_3\zeta_0 z^3 \tag{33}$$

with

$$z = r_0 q/\tau \tag{34}$$

$$\eta_0 = \sigma_0\tau \tag{35a}$$

$$\zeta_0 = \varepsilon_0\tau^2 \tag{35b}$$

$$\chi_0 = \rho_0\tau^2 \tag{35c}$$

and

$$\begin{aligned} \phi(z) &= c_3\left[(\mathbf{r}_0 \cdot \mathbf{P}_0)\frac{\tau^2}{r_0^2}\right]z^3 + c_4\left\{[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')]\frac{\tau^3}{r_0^3}\right\}z^4 \\ &+ c_5\left\{\frac{2r_0'\tau^4}{r_0^5}[3(\mathbf{r}_0' \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \mathbf{P}_0')] + \frac{\tau^4}{r_0^4}[3(\mathbf{r}'' \cdot \mathbf{P}_0) \right. \\ &\left. + 4(\mathbf{r}_0' \cdot \mathbf{P}_0') + (\mathbf{r}_0 \cdot \mathbf{P}_0'')]\right\}z^5 + c_6\left\{\frac{3r_0''\tau^5}{r_0^6}[3(\mathbf{r}_0' \cdot \mathbf{P}_0) \right. \\ &+ (\mathbf{r}_0 \cdot \mathbf{P}_0')] + \frac{3r_0'\tau^5}{r_0^6}[3(\mathbf{r}_0'' \cdot \mathbf{P}_0) + 4(\mathbf{r}_0' \cdot \mathbf{P}_0') + (\mathbf{r}_0 \cdot \mathbf{P}_0'')] \\ &+ \frac{\tau^5}{r_0^5}[3(\mathbf{r}_0''' \cdot \mathbf{P}_0) + 7(\mathbf{r}_0'' \cdot \mathbf{P}_0') + 5(\mathbf{r}_0' \cdot \mathbf{P}_0'') \\ &\left. + (\mathbf{r}_0 \cdot \mathbf{P}_0''')]\right\}z^6 + \cdots \end{aligned} \tag{36}$$

The argument of the functions c_n is

$$\lambda^2 = \alpha_0^2 q^2 = \left[\chi_0 - \frac{(\mathbf{r}_0 \cdot \mathbf{P}_0)}{r_0^2}\tau^2\right]z^2 \tag{37}$$

The function $\phi(z)$ may be written in a simpler form

$$\phi(z) = \tau^2 z^3\{c_3 a_1 + c_4 a_2\tau z + c_5 a_3\tau^2 z^2 + c_6 a_4\tau^3 z^3 + \cdots\} \tag{38}$$

with

$$a_1 = \frac{(\mathbf{r}_0 \cdot \mathbf{P}_0)}{r_0^2} \tag{39a}$$

$$a_2 = \frac{3(\dot{\mathbf{r}}_0 \cdot \mathbf{P}_0) + (\mathbf{r}_0 \cdot \dot{\mathbf{P}}_0)}{r_0^2} \tag{39b}$$

$$a_3 = 3\sigma_0 a_2 - 3\mu_0 a_1 + b \tag{39c}$$

$$a_4 = [4(\varepsilon_0 + a_1) + 3\sigma_0^2 - \mu_0]a_2 + 6\sigma_0 b + c - 3\mu_0\left[3\sigma_0 a_1 + \frac{2(\mathbf{r}_0 \cdot \dot{\mathbf{P}}_0)}{r_0^2}\right] \tag{39d}$$

where

$$b = \frac{3(\mathbf{P}_0 \cdot \mathbf{P}_0) + 4(\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{P}}_0) + (\mathbf{r}_0 \cdot \ddot{\mathbf{P}}_0)}{r_0^2} \tag{39e}$$

$$c = \frac{10(\mathbf{P}_0 \cdot \dot{\mathbf{P}}_0) + 5(\dot{\mathbf{r}}_0 \cdot \ddot{\mathbf{P}}_0) + (\mathbf{r}_0 \cdot \dddot{\mathbf{P}}_0)}{r_0^2} \tag{39f}$$

In these expressions we changed back to derivatives with respect to τ for convenience in later calculations.

Equation (33) is the main equation to be used in the perturbed problem of two bodies.

VI. Calculation of Ephemerides in the Perturbed Problem of Two Bodies

The method developed by Stumpff is based on the formulas [Eqs. (3) and (4)] that correspond to the planar motion in absence of perturbations.

In the perturbed problem, equations like Eqs. (3) and (4) are possible only if the perturbing force lies always in the plane defined by $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$. When such is not the case, we tried to solve the problem by approximating the real motion by successively approximating planar motions. It turned out that the method was rather complicated and its precision not quite satisfactory. In consequence, we solved the problem by adding in Eqs. (3) and (4) a third term proportional to the initial angular moment $\mathbf{g}_0 = \mathbf{r}_0 \times \dot{\mathbf{r}}_0$, that is

$$\mathbf{r} = F\mathbf{r}_0 + G\dot{\mathbf{r}}_0 + H\mathbf{g}_0 \tag{40}$$

$$\dot{r} = \dot{F}r_0 + \dot{G}\dot{r}_0 + \dot{H}g_0 \tag{41}$$

where F, G, H, $\dot{F}$, $\dot{G}$, and $\dot{H}$ are functions of the solution of the main equation [Eq. (33)].

To obtain these functions, we proceeded as follows:

Let us indicate with P_{r_0}, P_{v_0}, and P_{g_0} the components of the perturbation P along the directions of r_0, $\dot{r}_0$, and g_0, respectively. The differential equation of motion in vectorial form is

$$\ddot{r} = -\mu r + P \tag{42}$$

where μ is defined by Eq. (10a). By virtue of Eqs. (40) and (41) we obtain

$$\ddot{F}r_0 + \ddot{G}\dot{r}_0 + \ddot{H}g_0 = -\mu(Fr_0 + G\dot{r}_0 + Hg_0) + P \tag{43}$$

from which we derived

$$\ddot{F} + \mu F = P_{r_0} \tag{44}$$

$$\ddot{G} + \mu G = P_{v_0} \tag{45}$$

$$\ddot{H} + \mu H = P_{g_0} \tag{46}$$

Let us apply in Eq. (44) the regularizing transformation [Eq. (18)] and, by virtue of Eq. (13b), we obtain, after some simple operations,

$$F''' + \alpha^2 F' = 3r^3\sigma P_{r_0} + r^2 P'_{r_0} \tag{47}$$

where the primes indicate derivatives with respect to q and α as given by Eq. (21).

Recalling the definition of Stumpff's functions given in Sec. IV, Taylor's expansion of $F(q)$ at $q = 0$ becomes

$$\begin{aligned} F(q) = F_0 &+ F'_0(c_1 + \alpha_0^2 q^2 c_3)q + F''_0(c_2 + \alpha_0^2 q^2 c_4)q^2 \\ &+ F'''_0(c_3 + \alpha_0^2 q^2 c_5)q^3 + F_0^{IV}(c_4 + \alpha_0^2 q^2 c_6)q^4 \\ &+ F_0^{V}(c_5 + \alpha_0^2 q^2 c_7)q^5 + \cdots \end{aligned} \tag{48}$$

with

$$c_n = c_n(\alpha_0^2 q^2) = c_n\{[\rho_0 r_0^2 - (r_0 \cdot P_0)]q^2\} \tag{49}$$

Expansion (48) may be written in the form

$$\begin{aligned} F(q) = F_0 &+ c_1 F'_0 q + c_2 F''_0 q^2 + c_3(F'''_0 + \alpha_0^2 F'_0)q^3 \\ &+ c_4(F_0^{IV} + \alpha_0^2 F''_0)q^4 + c_5(F_0^{V} + \alpha_0^2 F'''_0)q^5 + \cdots \end{aligned} \tag{50}$$

In Eq. (50), the expressions in parentheses may be calculated from Eq. (47) and its successive derivatives with respect to q.

Furthermore,

$$F_0 = 1, \qquad F'_0 = r_0, \qquad \dot{F}_0 = 0 \tag{51a}$$

$$F''_0 = r_0^2[-\mu_0 + (P_{r_0})_0] \tag{51b}$$

where $(P_{r_0})_0$ is the component of the perturbation along the direction of r_0 and evaluated at t_0. Equations (51a) result from Eqs. (40) and (41) and Eqs. (51b) results from the fact that

$$F'' = r^2(\ddot{F} + \sigma\dot{F}) \tag{52a}$$

$$\ddot{F} = -\mu F + P_{r_0} \tag{52b}$$

The function F may be expressed as a function of z defined by Eq. (34):

$$F(z) = 1 - c_2\xi_0 z^2 + L(z) \tag{53}$$

where

$$\xi_0 = \mu_0 \tau^2$$

$$L(z) = \tau^2 z^2\{c_2 f_1 + c_3 f_2 \tau z + c_4 f_3 \tau^2 z^2 + c_5 f_4 \tau^3 z^3 + \cdots\} \tag{54}$$

The coefficients f_i depend on the component P_{r_0} of the perturbation, its derivatives, and the local invariables at the instant t_0. They can be evaluated by the following set of formulas:

$$f_1 = (P_{r_0})_0 \tag{55a}$$

$$f_2 = 3\sigma_0(P_{r_0})_0 + \frac{(P'_{r_0})_0}{r_0} \tag{55b}$$

$$f_3 = 3\{\varepsilon_0 + \sigma_0^2 + a_1\}(P_{r_0})_0 + \frac{5\sigma_0}{r_0}(P'_{r_0})_0 + \frac{(P''_{r_0})_0}{r^2} \tag{55c}$$

$$\begin{aligned} f_4 = {}& [5(P_{r_0})_0 - 2\mu_0]a_2 + 3\sigma_0[4(\varepsilon_0 + a_1) - \mu_0](P_{r_0})_0 \\ &+ \frac{8}{r_0}(\varepsilon_0 + \sigma_0^2 + a_1)(P'_{r_0})_0 + \frac{7\sigma_0}{r_0^2}(P''_{r_0})_0 + \mu_0(P'''_{r_0})_0 \end{aligned} \tag{55d}$$

In Eq. (54), the coefficients c_n are Stumpff's functions with argument

$$\left[\chi_0 - \frac{(r_0 \cdot P_0)\tau^2}{r_0^2}\right]z^2$$

To obtain $\dot{F}(z)$ we have, by virtue of Eq. (18)

$$\dot{F} = F'/r \tag{56}$$

and consequently

$$\dot{F}(z) = -c_1\frac{\mu_0 \tau z}{\Delta} + E(z) \tag{57}$$

where

$$\Delta = \frac{r}{r_0} = 1 + c_1\eta_0 z + c_2\zeta_0 z^2 + \phi^*(z) \tag{58}$$

$$\phi^*(z) = \tau^2 z^2\{c_2 a_1 + c_3 a_2 \tau z + c_4 a_3 \tau^2 z^2 + c_5 a_4 \tau^3 z^3 + \cdots\} \tag{59}$$

$$E(z) = \frac{\tau z}{\Delta}\{c_1 f_1 + c_2 f_2 \tau z + c_3 f_3 \tau^2 z^2 + c_4 f_4 \tau^3 z^3 + \cdots\} \tag{60}$$

Similarly, we have

$$G(z) = \tau z\{c_1 + c_2\eta_0 z\} + L^*(z) \tag{61}$$

with

$$L^*(z) = \tau^2 z^2\{c_2 g_1 + c_3 g_2 \tau z + c_4 g_3 \tau^2 z^2 + c_5 g_4 \tau^3 z^3 + \cdots\} \tag{62}$$

$$g_1 = (P_{v_0})_0 \tag{63a}$$

$$g_2 = 3\sigma_0(P_{v_0})_0 + \frac{(P'_{v_0})_0}{r_0} \tag{63b}$$

$$g_3 = a_2 + 3(\varepsilon_0 + \sigma_0^2 + a_1)(P_{v_0})_0 + \frac{5\sigma_0}{r_0}(P'_{v_0})_0 + \frac{(P''_{v_0})_0}{r_0^2} \tag{63c}$$

$$\begin{aligned} g_4 = {}& a_3 + 5a_2(P_{v_0})_0 + 3\sigma_0[4(\varepsilon_0 + a_1) - \mu_0](P_{v_0})_0 \\ &+ \frac{8}{r_0}(\varepsilon_0 + \sigma_0^2 + a_1)(P'_{v_0})_0 + \frac{7\sigma_0}{r_0^2}(P''_{v_0})_0 + \mu_0(P'''_{v_0})_0 \end{aligned} \tag{63d}$$

Similarly,

$$\dot{G}(z) = \left\{\frac{c_0 + c_1\eta_0 z}{\Delta}\right\} + E^*(z) \tag{64}$$

with

$$E^*(z) = \frac{\tau z}{\Delta}\{c_1 g_1 + c_2 g_2 \tau z + c_3 g_3 \tau^2 z^2 + c_4 g_4 \tau^3 z^3 + \cdots\} \tag{65}$$

Finally,

$$H(z) = \tau^2 z^2\{c_2 h_1 + c_3 h_2 \tau z + c_4 h_3 \tau^2 z^2 + c_5 h_4 \tau^3 z^3 + \cdots\} \tag{66}$$

$$\dot{H}(z) = \frac{\tau z}{\Delta}\{c_1 h_1 + c_2 h_2 \tau z + c_3 h_3 \tau^2 z^2 + c_4 h_4 \tau^3 z^3 + \cdots\} \tag{67}$$

with

$$h_1 = (P_{g_0})_0 \tag{68a}$$

$$h_2 = 3\sigma_0 (P_{g_0})_0 + \frac{(P'_{g_0})_0}{r_0} \tag{68b}$$

$$h_3 = 3(\varepsilon_0 + \sigma_0^2 + a_1)(P_{g_0})_0 + \frac{5\sigma_0}{r_0}(P'_{g_0})_0 + \frac{(P''_{g_0})_0}{r_0^2} \tag{68c}$$

$$h_4 = 5a_2(P_{g_0})_0 + 3\sigma_0[4(\varepsilon_0 + a_1) - \mu_0](P_{g_0})_0 + \frac{8}{r_0}(\varepsilon_0 + \sigma_0^2 + a_1)(P'_{g_0})_0 + \frac{7\sigma_0}{r_0^2}(P''_{g_0})_0 + \mu_0(P'''_{g_0}) \tag{68d}$$

VII. Summary of Formulas

A. Initial Data

Time: t_0; position and velocity of the perturbed body: $\boldsymbol{r}_0$, $\dot{\boldsymbol{r}}_0$; mass, position, and velocity of the perturbing body: m_p, $\boldsymbol{R}_0$, $\dot{\boldsymbol{R}}_0$.

B. Fundamental and Derived Invariables

$$\mu_0 = \frac{1}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)^{3/2}}$$

$$\sigma_0 = \frac{(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{r}}_0)}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)}$$

$$\omega_0 = \frac{(\dot{\boldsymbol{r}}_0 \cdot \dot{\boldsymbol{r}}_0)}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)}$$

$$\varepsilon_0 = \omega_0 - \mu_0$$

$$\rho_0 = \mu_0 - \varepsilon_0$$

from which are calculated

$$\xi_0 = \mu_0 \tau^2$$

$$\eta_0 = \sigma_0 \tau$$

$$\zeta_0 = \varepsilon_0 \tau^2$$

$$\chi_0 = \rho_0 \tau^2$$

with $\tau = k(t - t_0)$ where $k = 0.1720209895$ is the Gaussian gravitational constant.

C. Main Equation

$$z + c_2 \eta_0 z^2 + c_3 \zeta_0 z^3 = 1 - \phi(z)$$

where

$$\phi(z) = \tau^2 z^3\{c_3 a_1 + c_4 a_2 \tau z + c_5 a_3 \tau^2 z^2 + c_6 a_4 \tau^3 z^3 + \cdots\}$$

The coefficients a_i are calculated by Eq. (39), and c_i are Stumpff's functions defined in Sec. IV. The main equation is solved by a process of successive iterations.

D. Position and Velocity at Instant *t*

Calculate

$$\Delta(z) = 1 + c_1 \eta_0 z + c_2 \zeta_0 z^2 + \phi^*(z)$$

with

$$\phi^*(z) = \tau^2 z^2\{c_2 a_1 + c_3 a_2 \tau z + c_4 a_3 \tau^2 z^2 + c_5 a_4 \tau^3 z^3 + \cdots\}$$

$$F(z) = 1 - c_2 \xi_0 z^2 + L(z)$$

with

$$L(z) = \tau^2 z^2\{c_2 f_1 + c_3 f_2 \tau z + c_4 f_3 \tau^2 z^2 + c_5 f_4 \tau^3 z^3 + \cdots\}$$

$$G(z) = \tau z\{c_1 + c_2 \eta_0 z\} + L^*(z)$$

with

$$L^*(z) = \tau^2 z^2\{c_2 g_1 + c_3 g_2 \tau z + c_4 g_3 \tau^2 z^2 + c_5 g_4 \tau^3 z^3 + \cdots\}$$

$$H(z) = \tau^2 z^2\{c_2 h_1 + c_3 h_2 \tau z + c_4 h_3 \tau^2 z^2 + c_5 h_4 \tau^3 z^3 + \cdots\}$$

$$\dot{F}(z) = -c_1 \frac{\mu_0 \tau z}{\Delta} + E(z)$$

with

$$E(z) = \frac{\tau z}{\Delta}\{c_1 f_1 + c_2 f_2 \tau z + c_3 f_3 \tau^2 z^2 + c_4 f_4 \tau^3 z^3 + \cdots\}$$

$$\dot{G}(x) = \left\{\frac{c_0 + c_1 \eta_0 z}{\Delta}\right\} + E^*(z)$$

with

$$E^*(z) = \frac{\tau z}{\Delta}\{c_1 g_1 + c_2 g_2 \tau z + c_3 g_3 \tau^2 z^2 + c_4 g_4 \tau^3 z^3 + \cdots\}$$

$$\dot{H}(z) = \frac{\tau z}{\Delta}\{c_1 h_1 + c_2 h_2 \tau z + c_3 h_3 \tau^2 z^2 + c_4 h_4 \tau^3 z^3 + \cdots\}$$

and finally

$$\boldsymbol{r} = F(z)\boldsymbol{r}_0 + G(z)\dot{\boldsymbol{r}}_0 + H(z)\boldsymbol{g}_0$$

$$\dot{\boldsymbol{r}} = \dot{F}(z)\boldsymbol{r}_0 + \dot{G}(z)\dot{\boldsymbol{r}}_0 + \dot{H}(z)\boldsymbol{g}_0$$

E. Some Remarks

1) The coefficients f_i, g_i, and h_i depend on the local invariables and on the components of the vectors $\boldsymbol{P}_0$, $\dot{\boldsymbol{P}}_0$, $\ddot{\boldsymbol{P}}_0$, and $\dddot{\boldsymbol{P}}_0$ along the directions defined by the position, velocity, and angular momentum vectors at the initial instant t_0. The vector $\boldsymbol{P}_0$ can be calculated from Eq. (9) and its derivatives can be obtained either by analytical formulas or by standard formulas of numerical differentiation. (In spite of this addition to the computational effort, in actual examples that will be shown later, the method proved to be efficient as compared with the performance of a conventional method for the numerical integration of the differential equations of motion.)

2) For the numerical calculation of an ephemerides of the perturbed body corresponding to a set of successive instants $t_n = t_0 + nw$ $(n = 1,2,...)$ there are two alternative procedures: First, for the calculation of position and velocity corresponding to an instant t_k to use as initial data the results of the previous instant t_{k-1}. Second, to keep for all instants the same initial data corresponding to t_0. In the first case, $\tau = kw$ remains small and this assumes a fast convergence of

Table 1 Lagrangian problem: Differences between universal and exact solutions

	$w=0.5d$		$w=1.0d$		$w=5.0d$		$w=10.0d$	
$t-t_0$	Position	Velocity	Position	Velocity	Position	Velocity	Position	Velocity
50d	1.$D-15$	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-11$	1.$D-10$	1.$D-10$	1.$D-09$
100d	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-13$	1.$D-10$	1.$D-10$	1.$D-09$	1.$D-09$
150d	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-13$	1.$D-10$	1.$D-10$	1.$D-09$	1.$D-09$
200d	1.$D-14$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-10$	1.$D-10$	1.$D-09$	1.$D-09$

Table 2 Lagrangian problem: Differences between universal and exact solutions

	$e=0.00$		$e=0.05$		$e=0.15$		$e=0.5$		$e=0.75$		$e=0.9$	
$t-t_0$	Position	Velocity	Position	Velocity	Position	Velocity	Position	Velocity	Position	Velocity	Position	Velocity
50d	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-13$	1.$D-15$	1.$D-14$	1.$D-15$	1.$D-15$
100d	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-14$	1.$D-15$
150d	1.$D-13$	1.$D-12$	1.$D-13$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-14$	1.$D-14$
200d	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-14$
250d	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-14$
300d	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-14$
350d	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-14$
400d	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-12$	1.$D-13$	1.$D-13$	1.$D-13$	1.$D-14$	1.$D-14$	1.$D-13$	1.$D-14$	1.$D-14$

Table 3 Noncoplanar three-body problem: Differences between universal and Everhart's solutions

$t-t_0$	$I=0.01$	$I=0.1$	$I=0.25$	$I=0.5$	$I=1.0$	$I=1.5$
50d	1.$D-11$	1.$D-11$	1.$D-11$	1.$D-11$	1.$D-11$	1.$D-11$
100d	1.$D-11$	1.$D-11$	1.$D-11$	1.$D-10$	1.$D-11$	1.$D-11$
150d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$
200d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$
250d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$
300d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$
350d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$
400d	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$	1.$D-10$

Stumpff's series. On the other hand, it requires the recalculation at each step of new values for the local invariables and for the perturbation $\boldsymbol{P}$ and its derivatives. In the second case, the mentioned recalculation is unnecessary, but $\tau = nkw$ may become large and the accuracy of the results may be degraded. This alternative is faster and it may be used when the number of steps n and the required precision are not too high.

VIII. Examples

A. Coplanar Case

For this case, our examples refer to the Lagrangian solution of the three-body problem. The barycentric position of each body is represented at an instant t by vectors $\boldsymbol{R}_i$ $(i = 1,2,3)$ such that $\boldsymbol{R}_i(t) = \rho \boldsymbol{R}(t_0)$. Here $\boldsymbol{R}_i(t_0)$ are the same vectors at the initial time and ρ is the modulus of the radius vector of a "virtual" two-body problem characterized by given values of the eccentricity e and pericenter distance q.[5,6]

The magnitude of the closest approach of the three bodies is given by

$$|\boldsymbol{R}_i - \boldsymbol{R}_j| = q\,|\boldsymbol{R}_i(t_0) - \boldsymbol{R}_j(t_0)|$$

At the initial time it is assumed that the three bodies are located on the vertices of an equilateral triangle with unit sides. The triangular configuration is preserved, although the sides may vary periodically. If the eccentricity e and the pericenter distance q of the virtual ellipse are large and small, respectively, the closest distance between the three bodies may become very small.

In a numerical integration of the differential equations of motion, it would then be necessary to reduce considerably the step size of the process in order to keep the global errors of the results within reasonable limits. The method we are proposing is, in fact, rather insensitive to such difficulties.

In the examples that follow we adopted the masses $m_1 = 1$, $m_2 = 4.5205283D-04$, and $m_3 = 0$, corresponding to the central, perturbing, and perturbed bodies, respectively, and the numerical procedure was that indicated by the first alternative procedure in Sec. VII.E.

Example 1

We adopted in this case $e = 0.15$ and $q = 1$. Table 1 shows the differences between the results of our method and the exact results of the Lagrangian analytical solution[5] as a function of the time elapsed (in days) from the initial instant t_0 for several values of the adopted step size w.

Example 2

Table 2 shows results similar to those of Table 1, corresponding to several values adopted for the eccentricity of the virtual ellipse of the Lagrangian analytical solution. A large eccentricity results in a closer approach between the attracting bodies in which case the regularization of the problem comes actually into play. The procedure was the same as that adopted in Example 1 and the step size was $w = 1d$.

B. Noncoplanar Case

Here we applied the basic formulas [Eqs. (40) and (41)] in the form described in Sec. VII. The results obtained were

Table 4 Noncoplanar three-body problem: Differences between universal and Everhart's solutions

$t - t_0$	$w = 1.0d$	$w = 5.0d$	$w = 10.0d$	$w = 20.0d$
50d	1.D − 11	1.D − 11	1.D − 10	1.D − 09
100d	1.D − 11	1.D − 10	1.D − 09	1.D − 08
150d	1.D − 10	1.D − 10	1.D − 09	1.D − 08
200d	1.D − 10	1.D − 10	1.D − 09	1.D − 08
250d	1.D − 10	1.D − 10	1.D − 09	1.D − 08
300d	1.D − 10	1.D − 10	1.D − 09	1.D − 08
350d	1.D − 10	1.D − 10	1.D − 09	1.D − 08
400d	1.D − 10	1.D − 10	1.D − 09	1.D − 08

Table 5 Noncoplanar three-body problem: Differences between universal and Everhart's solutions

$t - t_0$	$I = 0.1$	$I = 0.5$	$I = 1.0$
50d	1.D − 14	1.D − 13	1.D − 13
100d	1.D − 14	1.D − 12	1.D − 12
150d	1.D − 14	1.D − 12	1.D − 12
200d	1.D − 13	1.D − 12	1.D − 12
250d	1.D − 13	1.D − 12	1.D − 12
300d	1.D − 13	1.D − 12	1.D − 12
350d	1.D − 13	1.D − 12	1.D − 12
400d	1.D − 13	1.D − 12	1.D − 12

Table 6 Lagrangian problem: Differences between universal and exact solutions

$t - t_0$	$w = 1.0d$	$w = 5.0d$	$w = 10.0d$	$w = 15.0d$	$w = 20.0d$
50d	1.D − 15	1.D − 12	1.D − 11	1.D − 10	1.D − 10
100d	1.D − 14	1.D − 11	1.D − 11	1.D − 10	1.D − 09
150d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09
200d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09
250d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09
300d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09
350d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09
400d	1.D − 14	1.D − 12	1.D − 11	1.D − 10	1.D − 09

Table 7 Noncoplanar three-body problem: Ratios of computer times for Everhart's and universal solutions

N	w	T_E/T_S
25	1.0	2.5
50	1.0	2.6
100	1.0	2.5
5	10.0	2.0
10	10.0	2.1
10	20.0	1.9
10	40.0	1.9

compared to the results of a numerical integration of the equations of motion performed with a method developed by Everhart.[7] This is a multistep method of variable order that has proven to be rather accurate and efficient as compared to other leading methods. In the following examples, we used Everhart's method of order 10 and adopted small step sizes to make the comparison of its results with those of our method meaningful.

Example 3

The masses of the bodies were the same as the previous examples. The eccentricity of the unperturbed orbit was $e = 0.15$, and the step size for the application of ours and Everhart's methods were 1 and 10 days, respectively. The objective of this example is to show the performance of our method for several values of the inclination of the plane of motion of the perturbing body against the plane of motion of the unperturbed orbit of the small body. Table 3 shows the differences between the results of both methods corresponding to several values of the inclination I expressed in radians.

Example 4

Table 4 shows the results corresponding to an eccentricity $e = 0.15$ and an inclination $I = 0.1$ rad, for several values of the step size w adopted in the application of our method. In the application of Everhart's method the step size was $w = 10$ days.

It may be noticed that with a step size $w = 20$ days (double the step size used in the application of Everhart's method) our results were coincident up to seven digits.

Example 5

This example tests the performance of our method for a high eccentricity $e = 0.9$. Table 5 shows the results corresponding to several values of the inclination I. The step sizes in both ours and Everhart's methods were $w = 1$ day.

Example 6

In this case the inclination was $I = 0$ (coplanar case) with a high eccentricity $e = 0.9$ and several values of the step size. The comparison was made against Lagrange's analytical solution.

Example 7

In this example we compare the machine time required for the application of Everhart's and our method under similar conditions. The problem solved was that of noncoplanar motion with eccentricity $e = 0.15$ and inclination $I = 0.1$ rad.

In Table 7 the first column is the number of steps and the second column the step size. In the third column we give the ratios between the times required by the corresponding applications of Everhart's and our method. It may be concluded that the efficiency of our method is about or higher than double. We used, in all cases, the same step size for both methods.

IX. Conclusions

We have developed a universal formulation for the treatment of the perturbed problem of two bodies, furnishing an efficient method for the calculation of ephemerides starting from initial values of position and velocity.

From our applications one may conclude that the proposed method may give a high degree of precision even in cases of large eccentricities and inclination. In such cases, the usual methods for the numerical integration of the equations of motion require a significant reduction of the step sizes, whereas our method is rather insensitive to such difficulties. In other words, the effect of the regularizing transformation is, in the usual methods, an increase of the computational effort inversely proportional to the reduced step sizes, whereas in our method it is not. On the other hand, a large inclination justifies our generalization of the problem as compared to the planar case treated in Stumpff's theory.

The perturbed two-body problem treated in this article is in fact an abstract case, although in many practical applications it may be sufficiently close to real cases like the motion of an asteroid where the most important perturbation is due to the gravitational attraction of Jupiter. A subject for future work is the extension of this method to the general case of the N-body problem as well as the theoretical study of its stability.

Finally, our method enjoys the practical advantages of flexibility and simplicity of all single-step methods.

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